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Mathematics News Letter

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To mathematics in general, to the following causes in particular is this journal dedicated: (1) the common problems of grade, high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of M. A. of A. and N. C. of T. of M. projects.

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ETHICS AND MATHEMATICS

Mathematics of itself cannot be said to have an ethical quality, such quality being resident only in human behavior. But evidence seems to favor the doctrine that the long pursuit of mathematical study, or, indeed, of any study of a strictly scientific nature, has the effect of building into the mind a high degree of honesty and candor in thinking, whether the thinking be about so-called non-mathematical matters or matters strictly mathematical. The destructive effects of an error—either logical or quantitative—in any mathematical process are most vividly realized by the professional mathematical worker. They belong to the kind of effects which, like a fire at night, must be seen and known of all. An error in theology, in political science, in economics or any of the so-called social sciences, has not nearly the same degree of self-advertising property that mathematical error has. It is indeed violently and loudly explosive. Thus, the mathematician early learns that dishonest and deceptive thinking in his field is paradoxical. If he is ever to attain his principal objective, namely, mathematical truth and its sound

applications, he **must** deal with his materials with absolute **honesty**. Camouflage is fatal. He cannot dare to say I **know** when he does not know. When he errs, he must confess.

But, there are those who say: one may be honest in his mathematics and lack honesty in other respects. This is undoubtedly true. On the other hand, if there have been many years of thorough habituation of a mind to the use of honest judgment in distingushing logical truth and falsehood, the use of a standard that is satisfied with nothing less than all the truth, and a type of thought that squares one hundred per cent with accepted hypotheses, such a mind will carry over the same attitudes and ideals and standards into any field where it has hard thinking to do—**because it will have no other substitutes for them that can come out of its experience.** Certainly, they cannot issue from **another's** experience.

One has only to observe in order to **see** in the mental habits of the typical mathematician of wide experience evidence that these things are true: Caution in the making of fact statements, absence of dogmatizing on matters not subject to dogma, assurance only when there is scientific certainty, scrupulousness in the placing of quotation marks where such marks are due, scorn to assume a credit not deserved.

These are a few of the carried-over values. —*S. T. S.*

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THE SUPREME NEED IN MATHEMATICS TEACHING

In a word, it is an immeasurably greater degree of **preparedness** for freshman college work. Readers of the News Letter will recall the startling statistics indexing the lack of such preparedness which were furnished them several months ago by Professor Stone, of one of the New Jersey State Teachers' colleges. It will be remembered that Mr. Stone's examination questions were of a very elementary type taken from the first year of high school algebra, and that they were submitted to graduates of scores of high schools, many of them honor graduates.

So far as we know, the Mathematics News Letter is a pioneer journal in respect to the particular function it has dared to assume, namely, the function of representing the correlated ef-

forts of grade, secondary and college mathematics teachers to work out, as far as practicable, correctives for this general condition of unpreparedness.

A primary essential to even a partial success of these efforts is an unconcealed and frank recognition of the basic facts in the case. An honest, if necessary, even a brutal, analysis of the case for the purpose of accurately allocating the individual or collective responsibility for the situation should be made, and the results of such analysis used in a scheme of constructive reform if possible.

The News Letter invites, urges, all professonally-minded mathematics teachers to make most serious cooperative effort to effect, first, a correct diagnosis, second, a constructive scheme for bettering the situation. The invitation is to the grade, the high school, or the college worker.

In the face of the fact, that a demand for technical mathematics is steadily increasing in every field of human endeavor where exactness of method is required, and the further fact that fields which heretofore have made but slight use of mathematics as an instrument of efficiency, are calling in increasing numbers for its use-in the face of such facts, it is a puzzing and serious anomaly that confronts us, namely, the relatively slight attention paid to mathematics by the average school administration.

—*S. T. S.*

*COMPARISON OF TWO METHODS OF TEACHING PUPILS TO APPLY THE MECHANICS OF ARITHMETIC TO THE SOLUTION OF PROBLEMS.

By ALMA WILL
La Salle School, New Orleans, La.

We teachers have had emphasized in our training and find listed again and again in our books on arithmetic pedagogy—two particular objectives for teaching arithmetic. One is—skill

*Read at the New Orleans Institute of Teachers, September 9, 1930.

in computation; the other, power to solve every-day problems within one's experience.

In reference to skill in computation, Dr. Stone states that "each formal process should become to the pupil a machine. Automatic control of this machine as such should be the ultimate aim." The problem presents far more difficulties than the example in pure calculation because of the various elements which go toward its composition. It is well to bear in mind the distinguishing difference between an example and a problem. An example is an indicated solution. Its mathematical symbols tell the pupil what to do; whether to add, subtract, multiply or divide. "Klapper defines a problem as a situation coming naturally into the life of an individual and capable of arousing effort for its solution." The pupil must first decide what operations are to be performed and what their sequence should be before proceeding to the calculations.

Another authority gives us this information. "Chronologically, problems must have preceded examples. The race was confronted with a multitude of interesting problems long before it invented a mathematical symbolism to express number relations. Examples thus became devices for fixing the identical elements in problems". It is logical then to have problems both precede and follow examples. Later, the formal phases should be abstracted for memorization, and then again used in problems to insure their correct usage." The same authority further states: "In spite of the close relationship existing between examples and problems, they are frequently sharply separated and taught as if neither has much dependence upon the other." So we may ask ourselves: Which study is a preparation for the other? Which should be taught first? Will results be different if one is given preference over the other in order of teaching?

In answer to these questions, we find an article written by Carleton W. Washburne, Superintendent of Schools, Winnetka, Ill., in the Elementary School Journal of June, 1927, citing an experiment and its substance which endeavors to find an answer to these same questions.

The heading is worded thus: Will children acquire the greater ability to apply the fundamental processes in arithmetic to the solution of verbal problems if the processes are introduced

through situations within the children's experience and are applied throughout the period of practice directly to the solution of problems than if the processes are learned and practiced without reference to concrete situations and later applied to the solution of problems?

The experiment was handled by The Committee of Seven of the Superintendents' and Principals' Association of Northern Illinois which directed its attention in 1925-26, the fourth year of its research work, to a study of this comparison. The particular reason for this study arose as a result of the constant complaint that children have difficulty with the mechanics of arithmetic as applied to the solution of problems.

Besides causes already known which apparently seemed not to have offered a solution, the committee judged that another source of trouble might lie in the fact that many schools teach the mechanical processes quite apart from their practical applications. Would a more intimate relating of mechanics and practical problems from the beginning make easier the transition from mere ability to subtract, divide, or find the per cent one number is of another, for instance, to facility in the process of solving verbal problems?

In order to be able to answer this question the Committee secured the co-operation of superintendents and principals in 16 cities in northern Illinois. Forty-one of their teachers were chosen to try out the two methods with parallel classes.

The teachers received full instructions as to the method and procedure. After the completion of the study, all results and papers were turned over to the committee for final tabulation and summary.

In the sixth or seventh—Case II of percentage (finding the per cent one number is of another).

In each grade, the children were given an intelligence test, a problem-solving test in the nearest related process; which was addition facts in the second, short division in the fourth, and Case I in percentage in the sixth or seventh; also a test in the mechanics of this last learned process.

It was necessary that the teacher divide her class into equivalent groups. The method consisted essentially of arranging the children's names in their rank order as determined by the

principal test (problem-solving), dividing the group into four sections—highest quarter, second quarter, third quarter, and lowest quarter, then splitting each quarter into two parts so that the average of each part matched that of the other in (a) problem-solving ability, (b) ability in arithmetic mechanics, (c) mental age, (d) chronological age and (e) general ability to work as judged by the teacher.

Thus, there were eight subgroups of children—four subgroups ranking from bright to low to be taught one way (Group I) and four to be taught the other way (Group 2).

Group I was to be taught a number process through the use of verbal problems and with constant application to problems.

Group II was to be taught the same number process without regard to problems or concrete situations, until the mechanics were fairly well mastered, and then it was to concentrate on problem-solving.

Both groups were taught by the same teacher. When one group received oral instruction, the other group was out of the room.

The group taught first during the first part of the experiment was taught second during the second part.

Both groups devoted the same amount of time to the work each day for six weeks. Both used the same materials of instruction. Both used the same problems. Both took the same same tests. In fact, all factors were kept the same except the method and order of presentation of new material and problems.

Group I, using the 'problem method" made use of problems which required the use of the process to be taught. It was necessary to select problems which the pupils would desire to solve.

The most difficult part of the Committee's work lay in securing problems which would (1) involve familiar situations, (2) appeal to the pupil as worth solving, and (3) arouse in the pupil a desire to learn the process involved and to overcome the difficulties met in mastering it.

The committee recognized the fact that in order to interest all pupils many problems of different types would be needed—some for the teaching of the new process and later others for review and retesting purposes. So the committee prepared for

each phase of the experiment approximately 100 problems which were both real and within the pupils' range of experience and interest. It is important to note that the problems were not all equally good, in fact, some were poor.

However, the committee thought theirs better than those usually found in text-books.

Right here, our attention is called to the fact that this collection of problems was one of the valuable results of the experiment. Just try to prepare a few practical problems in long division with 2-place divisors and quotients of three digits about familiar situations interesting to fourth grade children, and see what effort it will require.

The instructions for teaching the groups by the problem method may be summarized as follows:

(1) Read through with the class two or three problems involving the new process in an early stage of difficulty.

Call the pupils' attention to the situations involved.

Ask whether any of them have actually encountered similar situations. Help them to see the importance of learning how to solve the problems.

(2) Show the class how to solve one of the problems. Work it out step by step and show the form and notation to be used clearly and fully. Repeat with the second problem.

(3) Let the pupils try the third problem. Then work it out with them on the board. Correct any errors and explain any difficulties noted in the work of individuals.

(4) Use the rest of the period for practice in working abstract examples of the same early stage of difficulty.

(5) On the second day review the problems solved the day before. Present a new problem to see whether the pupils can solve it. Have a pupil explain it to the class or demonstrate the solution as on the first day.

Let the class use additional drill materials until all are ready for the next step. The class may need one or two days of practice before it is ready for the same process with an added difficulty.

(6) Present two problems which involve a new difficulty. Solve them with the class. Bring out the new difficulty very clearly and show how it is overcome.

Give the class another problem of the same difficulty and work it out with the pupils if necessary. Spend the rest of the time on drill materials which involve the new difficulty.

(7) The next day review both the simple process and the new difficulty of the day before.

Give the class one or two problems which involve another difficulty and again explain the method of solution to the class or give individual help. Make use of drill materials involving this step.

Summarized, teaching Group I consisted of introducing the process being taught in its simplest form and each new step through problems; drilling the facts taught with problems, and using problems in tests and reviews.

This plan was followed for four weeks. The last two weeks were a preparation for the final test. It was spent reviewing everything which had been taught during the first four weeks, and also the processes which had been taught before the experiment began.

The method used in Group II consisted in simply plunging into the new process without assigning any reason and without giving any verbal problems. The mechanics of the process (or the facts) were taught independently of any concrete support. This was done for four weeks. The last two weeks were devoted to problems using the mechanics taught abstractly.

The problems used were the same as those used in Method I. At the end of the six-week period both groups were given a problem-solving test in the newly-learned process and a test in the mechanics of the process.

Unless one examines statistical tables himself, he will not be able to fully appreciate their contents.

Both means and medians were computed but, since there was, as a rule, no significant difference between the two sets of figures, the means only are given.

The problem-solving tests were scored in two ways, for correctness in the choice of process to be used and and for mechanical accuracy in the use of the correct process.

The record for 480 children in the 6th and 7th Grades who were taught Case II of Percentage likewise shows no appreciable

difference in results whether Method I or II was used.

A comparison of the three tables shows that there was no greater difference in accomplishment after the experiment than before it. Children seemed to learn both the mechanics and problem-solving equally well either way. Superintendent Washburne concludes his report thus:

"The Committee of Seven concludes, therefore, that teaching the mechanics of arithmetic—facts and processes—by themselves first and then applying them to the solution of practical problems does not lead to difficulty in making practical application of the mechanics to the solution of problems.

A combination of thorough training in the mechanics of arithmetic and thorough training in the use of the mechanics in solving practical problems produces good results. This is true regardless of whether or not the mechanics are introduced through problems and constantly used in practical problems while they are being learned. The mechanics of arithmetic may be taught thoroughly and then applied to practical problems, or the two types of teaching may be intimately related throughout the teaching process with equal efficacy."

It might prove worth while to consider a few points in our own daily teaching experience. Some of us have found it preferable in some instances to teach mechanics first and then apply to problems. For instance—In the first case of percentage, percentage like 3.54%, ½%, 27.6% besides giving them some difficulty when changing to decimals invariably are handled incorrectly in problems. By such abstract drills as 3.54% of 735, ½% of 250, 27.6% of 782 the pupils learn to handle them abstractly but yet not linked up with a problem.

Followed by simple 1st. case problems, the task is about 50% already accomplished.

Perhaps, someone else is more successful in handling these forms in problems immediately.

Square root can be taught more successfully by handling it abstractly first, and then, when well known, introducing problems where its application is necessary. Many problems involving square root require such serious thinking on the part of a student, that it is best that the mechanical side be already well mastered.

The exact method resembles long division to them and they learn it without questioning. By the time they begin to want to know—"what are we going to do with this"—they are well on their way to the mechanical handling of it.

In teaching stocks and bonds, the problem method is best, because it is the subject matter that is of the greatest importance; the mechanical side requiring no special new teaching.

We have all found that the individual child can be approached sometimes with one method, sometimes with the other. Children to whom mechanical arithmetic appeals will learn the mechanical perfectly and perhaps never be able to apply it to a problem. Why deprive him of this much knowledge, even if he never gets the other? Other children can learn nothing in arithmetic unless concretely illustrated. If possible, approach them with what appeals to them most.

In conclusion, Superintendent Washburn's article closes with a suggestion, which I am sure, meets with the approval of all of us, and is worth quoting. He states: "In the case of individual teachers, sometimes one method and sometimes the other shows superiority. Certainly there seems to be no justification whatever for requiring all teachers or children to give preference to either method. If a teacher has definite preference for one method and enthusiasm for that method, in all probability she will get better results for using it."

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THE TRANSFER VALUE OF MATHEMATICS

By LORRAINE SHELL
Okolona, Mississippi

Transfer, or training, concerns itself with the question as to how far training in one subject along one line influences other lines. How does the education which the pupil receives in school affect his subsequent thinking and conduct? Does an experience acquired today under a certain set of conditions carry over or transfer to a later period of life?

The old doctrine of transfer stated that mental power or training gained in one subject would spread or transfer to any

other subject. It made little difference in what connection one learned to reason, or to be self-possessed, or to stick to an unpleasant task. The powers thus developed would be applicable to any other situation and would operate in all ines of activity. The aim then in the teaching of mathematics was to develop the faculty of reasing.

Immediately following the scientific studies of transfer by Thorndike and Woodworth in 1901, there was an extreme reaction against this doctrine. Then we heard that all training is specific and that transfer is limited to situations that are identical with those in connection with which the training was secured.

To deny that training acquired by the study of any one subject can serve a pupil in thepursuit of any other, and, above all, that it can be of service to a man in meeting the emergencies of actual life is like saying that the "tackling dummy" is of no service to the football player or that the training of the child in well-devised bodily exercises has no bearing on subsequent life. In fact, if the theory advanced, that there is no transfer of training, be carried to its logical conclusion, there would be no justification for education at all, for schools are fundamentally the places of training for life by means of a discipline that, except for the specialist, will rarely in its precise form be met later.

The majority of psychologists now hold that transfer of training is an established fact and that it may be positive or negative. They agree that the amount of transfer from mathematics depends upon the method of teaching the subject, and that meanings, methods of attack, and attitudes and ideals are more transferable than skills and information.

The channels through which improvement is carried from one field to another have been designated by Thorndike as "Identical Elements". Transfer of training occurs when the same bonds are used in the second situation to the extent that the alteration in these particular connections affects the second response. In general, it occurs to the extent that the two responses use the same bonds—that there is an identity of some sort. These identities may be in content, method or procedure, or in terms of attitude and ideal.

First there may be an identity of content. For example, forming useful connections with triangles, percentage, and square root in this or that particular context can be of use in other contexts and therefore allow transfer of training. If the identity is that of method or procedure, the pupil must be able to generalize a conclusion from previous experiences and to recognize the fact or principle thus generalized in new situations. Judd is the foremost among those who believe that transfer takes place through generalization. Dr. Judd has formulated the theory of transfer which stresses the importance of a conscious recognition of the identical elements and the deliberate search for identical elements as a basis for generalizations. To be able to "add" or to "carry" or to have definite methods of meeting a new situation in mathematics or life is useful in other departments where the same method would serve.

In establishing skill in the use of these various procedures, two types of responses are needed. The learner must form connections of a positive nature, such as analyzing, criticizing according to standard, picking out the essential and so on, and he must also form connections of a negative character which will cause him to neglect certain tendencies. He learns not to accept the first idea offered, to neglect suggestions, to hurry or leave half-finished, to ignore interruptions, to prevent personal bias to influence criticism. This procedure is illustrated nicely in the study of Geometry.

The identity may be of still more general character and be in terms of attitude or ideal. Teachers of mathematics wish their pupils to be accurate, to develop an attitude of inquiry, to be persistent, to have a distaste for incomplete and slipshod work, and to get at the reason of things. Whether these traits transfer or remain effective only in connection with the situations in which they were learned depends largerly upon the teacher and her method of developing them. A child may be accurate in Arithmetic and yet show it nowhere else. He may be truthful in his geometry proofs, but lie to his mother. In order to increase the probability of this transfer when connections of method or attitudes are being formed, it should be made conscious and should be put into practice in several types of

situations. If the question of method, as an ideal by itself, apart from any particular subject is brought to the child's attention; if truth in proof of mathematics as an ideal, independent of context, is made conscious, it will be reacted to in a different situation, because it has become a "free idea" and therefore crystallized. Having freed the general from its particular setting, the learner is now able to put it in practice in other settings.

The teacher is, after all, the most important agent in producing the maximum amount of transfer. The most interesting subject from the standpoint of content and organization may be presented in a dry wooden manner, so that few values of any sort accrue to the student. On the other hand, if the teacher is alive, interested in his subject and in teaching; if he places emphasis upon topics of real importance and is careful to see that application of facts, principles and modes of procedure is made for his students to both school and out-of-school situations; his class will feel that they have gotten something from his course—that is, they will have been able to transfer their training.

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*THE VALUE OF THE STUDY OF SOLID GEOMETRY

J. T. HARWELL
Byrd High School, Shreveport, La.

Before I begin my talk, I want to say that I would consider myself a veritable ingrate, if I did not acknowledge in some way the splendid hospitality that the people of Cleveland have shown us since our arrival. I have never seen anything superior to it. It is strictly par excellent.

Nor am I unmindful of the compliments that have been given me individually. President Kethley introduced me to several persons as Doctor Harwell and presented me to the student body of the college as the principal of the C. E. Byrd High School, and to cap it all, as I walked into the lobby of the Cleveland building a little awhile ago, Miss Echols called out so loudly that all heard it, "Dean Hardin, there's a letter here for you".

*Read at the Cleveland, Miss., meeting of La.-Miss. Council.

I would indeed be callous, if I did not feel a little flattered, don't you think?

A few days ago, Dean Hardin drove up to my place and honked his horn. I went to see what the trouble was. He asked me if I could talk at the meeting in Cleveland. I replied, "Talk, talk! I wish I were paid for what talking I do. I would then have to pay an income tax." But since that is not the case, Uncle Sam does not worry me much about an income tax report.

As Dr. Maizlish and I came to this place of meeting a moment ago, we observed on the south end of this building an inscription which is apropos to my subject. It reads, "The State has said that only free men shall be educated, but God has decreed that only educated men shall be free." That sounds very much like our freedom bears a direct ratio to our education and I can find no fault with that. It is more or less axiomatic, I think.

We might begin by observing that the value of the study of Solid Geometry runs along several lines. There is, first, a disciplinary value. There was a time when men were said to be because of the eytmology of legal terms or the influence of Solon able to succeed at law, if they had mastered the Greek verb,—not or Iycrugus on the legal thought of the age, but for the mental training acquired through concentrated and determined effort. The pupil who, by gradual growth, can take the hypotheses of solid geometry and, by logical thinking, can reach the required conclusions is not so likely to go very far wrong in his thinking in politics and religion.

Then there is the practical value. Nowadays, pupils want to know what use they can make of every subject in high school. "Will I ever need this in life?" "What good is it anyway?" "Will I be able to make more money by knowing it?" They are decidedly of a practical turn of mind. Well, we live in three dimensions and everything conceived or constructed or consumed lends itself in some way to the matter of solid geometry. We lay the foundation of a house and plumb the framing by having it make right angles with two of the sills, knowing that a line perpendicular to two other lines at their point of inter-section is pependicular to the plane of these lines. The volume of excava-

tions or fills, the size of a piece of ice, the size of eggs or apples, or the cover of a baseball—all are geometric. Even the amount of food necessary to make a "square" meal may bear some peculair ratio to one's waistline.

There is, also, the aesthetic value of this study of solid geometry. The great masterpieces of art in sculpture and painting could hardly be appreciated without a knowledge of perspective, proportion, and symmetry in three dimensions. We have before us the painting of a battlefield. Here is a sword. A hundred yards away is a brokendown caisson. In the distance is a dismantled fort. The artist has depicted all this in a plane, but the appreciation comes from the sensing of a third dimension. Beauty of form in sculpture and architecture are absolutely dependent upon proportion in three dimensions.

Finally, a moral value may be gained from our study. The habit of expecting exactness, accuracy, or perfection, such as is exemplified in the cube, the cone, or the sphere, leads us to a like exactness in dealing with our fellow man. The American people are prone to be satisfied with approximations to such an extent that we are approximately honest in our lives,—we approach honesty as a limit and are therewith content. May we hope that, as teachers of mathematics, we may see in this exactness and beauty of life and character, realizing that the epitome of creation is man and that God geometrizes whether dealing with body, mind, or spirit.

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SEXTANT AND BI-SEXTANT TRIANGLES

By B. E. MITCHELL
Millsaps College, Jackson, Mississippi.

2. THEIR TRIGONOMETRY.

In the preceding issue of the News Letter we defined sextant and bi-sextant triangles as plane triangles having one of their angles equal to Sixty Degrees and One hundred Twenty Degrees respectively. These triangles partake, naturally, of the characteristics of both right triangles and oblique triangles and we shall draw now from one side of their kinship and now from another.

1. *Fundamental Relations.* If we choose for the triangle ABC the angle $C=60^\circ$ and the angle B the greatest of the angles, the cosine law applied to the c-elements yields:

(1) $c^2=a^2+b^2-2ab\cos 60^\circ \quad =a^2+b^2-ab,$

This is a relation among the sides which is the sextantal analogue of the Pythagorean relation in the right triangle, $a^2+b^2=c^2$.

For the bisextantal triangle, $C=120^\circ$, $a<b<c$:

(2) $c^2=a^2+b^2+ab.$

The cosine law applied to the other two pairs of elements making use of (1) and (2) in reduction:

(3) $\cos A=(2b-a)/2c$ and $\cos B=(2a-b)/2c$

for sextantal triangles and

(4) $\cos A=(2b+a)/2c$ and $\cos B=(2a+b)/2c$

for bisextantal triangles.

The sine law applied to each of these types of triangles yields:

$\sin A/a=\sin B/b=\sin C/c=\sqrt{3}/2c$, from which we deduce

(5) $\sin A=(\sqrt{3}/2)(a/c)$ and $\sin B=(\sqrt{3}/2)/(b/c)$.

We shall not consider the case of the bisextantal triangle beyond this point. The reader may easily develop the formulas for this triangle. Some of them are identical with those of the sextantal triangle and many differ from those of the sextantal triangle only in an algebraic sign.

2. *Auxiliary Relations.* The following formulas are obvious and from them we easily derive the remaining formulas of this paper.

(6) $\sin A+\sin B=(\sqrt{3}/2)(a+b)/c$

(7) $\sin A-\sin B=(\sqrt{3}/2)(a-b)/c$

(8) $\cos A+\cos B=(1/2)(a+b)/c$

(9) $\cos A-\cos B=(3/2)(a-b)/c$

(10) $\sqrt{3}\sin A+\cos A=(a+b)/c$

(11) $\sqrt{3}\sin A-\cos A=(2a-b)/c$

(12) $\sin A+\sqrt{3}\cos A=\sqrt{3}b/c$

(13) $\sin A-\sqrt{3}\cos A=\sqrt{3}(a-b)/c$

(14) $\sin A\sin B=3ab/4c^2$

(15) $\cos A\cos B=3ab/4c^2-1/2$

(16) $\sin A\cos A = \sqrt{3}(2ab - a^2)/4c^2$
(17) $\sin A\cos B = \sqrt{3}(2a^2 - ab)/4c^2$
(18) $\cos A\sin B = \sqrt{3}(2b^2 - ab)/4c^2$.

3. *The Complementary Relation Analogue.* The simplest relation between the two acute angles of a right triangle is the co-function relation, viz. $\sin A = \cos B$ and $\cos A = \sin B$. The corresponding analogue for sextantal triangles may be variously expressed by different combinations of formulas (6) to (13). The simplest, probably, are the ones furnished by (11) and (12), observing that $\sqrt{3}b/c = 2\sin B$ *and* $(2a - b)/c = 2\cos B$:

$$2\sin B = \sin A + \sqrt{3}\cos A$$

and

$$2\cos B = \sqrt{3}\sin A - \cos A.$$

These formulas may be derived directly by applying the addition theorems to the identity $B = 120^\circ - A$.

A formula involving the sines only arises from the definitions and the fundamental relation $a^2 + b^2 - ab = c^2$, viz.

$$\sin^2 A - \sin A\sin B + \sin^2 B = 3/4.$$

This, of course, is the analogue of $\sin^2 A + \cos^2 A = 1$ for right triangles since $\cos A = \sin B$. The corresponding formula involving cosines is

$$\cos^2 A + \cos A\cos B + \cos^2 B = 3/4.$$

4. *The Addition and Product Formulas.* The next formulas we consider are the addition formulas for A and B. Using (17) and (18), and then (15) and (14) we have:

$$\begin{aligned}\sqrt{3}\sin(A - B) &= 3(a^2 - b^2)/2c^2\\ &= 2(\sin^2 A - \sin^2 B)\\ &= 2(\cos^2 B - \cos^2 A).\end{aligned}$$

and

$$\begin{aligned}2\cos(A - B) &= (3ab - c^2)/c^2\\ &= 4\sin A\sin B - 1\\ &= 4\cos A\cos B + 1.\end{aligned}$$

The product or factor formulas of trigonometry, using (6) –(9), give

$$\begin{aligned}\sin(1/2)(A - B) &= \sqrt{3}(a - b)/2c\\ &= \sin A - \sin B\\ &= (\cos B - \cos A)\sqrt{3}\end{aligned}$$

and

$$\begin{aligned}\cos(1/2)(A - B) &= (a + b)/2c\\ &= (\sin A + \sin B)/\sqrt{3}\\ &= \cos A + \cos B.\end{aligned}$$

Directly by the use of the law of tangents or from the formulas immediately preceding we have:

$$\tan(1/2)(A-B)=\sqrt{3}(a-b)/(a+b)$$

5. *The Altitudes.* Let the altitudes upon the sides a, b, c be h_1, h_2, h_3 respectively. Then

$$2\triangle=ch_3=ab\sin 60^\circ=\sqrt{3}ab/2. \quad \text{Therefore}$$

$$2h_3=\sqrt{3}ab/c.$$

The other two altitudes are legs of quadrantal triangles having $C=60^\circ$ for one of the acute angles and the sides a and b as hypotenuses. Hence

$$2h_1=\sqrt{3}b \text{ and } 2h_2=\sqrt{3}a. \quad \text{Adding we have}$$

$$2c(h_1+h_2+h_3=\sqrt{3}(ab+bc+ac)$$

6. *The Medians.* Let the medians to the sides a, b, c, be m_1, m_2, m_3. If m_3 makes with c angles a *and* B, then the law of cosines applied to a and B gives

$$a^2=c^2/4+m_3^2-cm_3\cos a$$

and $$b^2=c^2/4+m_3^2-cm_3\cos B.$$

Since $$\cos a=-\cos B$$

$$a^2+b^2=c^2/2+m_3^2$$

or $$4m_3^2=2a^2+2b^2-c^2=a^2+b^2+ab.$$

The values of the other two medians are

$$4m_1^2=a^2+4b^2-2ab, \text{ and } 4m_2^2=4a^2+b^2-2ab.$$

Their difference yields the interesting result

$$4(m_2^2-m_1^2)=3(a^2-b^2)$$

In discussing this topic, The Trigonometry of the Sextantal and Bisextantal Triangles, we borrowed the definitions of the trigonometric functions from the right triangle. This is not necessary. These triangles have a trigonometry all their own and it is very interesting and highly instructive. In the near future we hope to have something to report in this subject.

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PARTIAL FRACTIONS

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In this paper will be given a powerful method for resolving fractions into partial fractions. Moreover the existence theorems

upon which the usual methods depend for their validity will be proved incidentally.

1. *Linear factors.* Let it be required to resolve the fraction $(x^4+x+1)/[(x-1)^3(x+2)^2]$ into partial fractions. Set $X=x-1$. We then find as the result of two divisions:

$$x^4+x+1=3+(x^3+x^2+x+2)X, \tag{1}$$
$$(x+2)^2=x^2+4x+4=9+(x+5)X.$$

We next "divide" the first of these by the second as shown below.

$$\begin{array}{r|l} & \quad 1/3 \\ \hline 9+(x+5)X & 3+(x^3+x^2+x+2)X \\ & 3+(\quad 1/3x+5/3)X \\ \hline & \quad (x^3+x^2+2/3x+V/3)X \end{array}$$

From this division we get

$$[3+(x^3+x^2+x+2)X]/[9+(x+5)X]$$
$$=(1/3)+[(x^3+x^2+2/3x+1/3)X]/[9+(x+5)X],$$

or, by (1),

$$(x^4+x+1)/(x+2)^2=1/3+[(x^3+x^2+2/3x+1/3)$$
$$(x-1)]/(x+2)^2.$$

On dividing both sides of this equation by $(x-1)^3$, we get

$$(x^4+x+1)/[(x-1)^3(x+2)^2]=1/[3(x-1)^3]$$
$$+(3x^3+3x^2+2x+1)/[3(x-1)^2(x+2)^2] \tag{2}$$

Next

$$3x^3+3x^2+2x+1=9+(3x^2+6x+8)X. \tag{3}$$

We now perform the following division:

$$\begin{array}{r|l} & 1 \\ \hline 9+(x+5)X & 9+(3x^2+6x+8)X \\ & 9+(\qquad\quad x+5)X \\ \hline & \quad (3x^2+5x+3)X \end{array}$$

From this we get

$$[9+(3x^2+5x+3)X]/[9+(x+5)X]=1+[(3x^2+5x+3)X]/$$
$$[9+(x+5)X]$$

or, by (1), (3),

$$(3x^3+3x^2+2x+1)/(x+2)^2=1+[(3x^2+5x+3)(x-1)]/$$
$$(x+2)^2$$

On dividing both members by $3(x-1)^2$, we get

$$(4)\qquad (3x^3+3x^2+2x+1)/[3(x-1)^2(x+2)^2]=1/[3(x-1)^2] + (3x^2+5x+3)/[3(x-1)(x+2)^2].$$

Next we find

$$(5)\qquad 3x^2+5x+3=11+(3x+8)X,$$

and perform the division

$$\begin{array}{l|l} & 11/9 \\ 9+(x+5)X & 11+(3x+8)X \\ & 11+(11/9x+55/9)X \\ \hline & (16/9x+17/9)X \end{array}$$

From this we get as above

$$(3x^2+5x+3)/(x+2)^2=11/9+[(16/9x+17/9)(x-1)] /(x+2)^2,$$

which in turn yields on dividing through by $3(x-1)$,

$$(6)\qquad (3x^2+5x+3)/[3(x-1)(x+2)^2]=11/[27(x-1)] + (16x+17)/[27(x+2)^2].$$

But finally

$$(7)\qquad 16x+17=-15+16(x+2),$$

from which we get on dividing through by $27(x+2)^2$,

$$(8)\qquad (16x+17)/[27(x+2)^2]=-5/[9(x+2)^2]+16/[27(x+2)].$$

From (2), (4), (6), (8) follows the final result

$$(x^4+x+1)/[(x+1)^3(x+2)^2]=1/[3(x-1)^3]+1/[3(x-)^2]. +11/[27(x-1)]-5/[9(x+2)^2]+16/[27(x+2)].$$

Let us now state the process in general terms. Let the fraction to be resolved into partial fraction be $V/(X^nV)$, where X, U, V are polynomials in x mutually prime and X is linear. By two synthetic divisions, we get

$$(9)\qquad U=a+U_1X,\quad V=b+V_1X.$$

Here both a and b are different from zero, since otherwise it would follow from (9) that X would not be prime to both U and V. We now perform the division

$$\begin{array}{l|l} & b/a \\ \hline a+U_1X & b+V_1X \\ & b+(bV_1/a)X \\ \hline & WX \end{array}$$

in which

(10) $\qquad W = V_1 - bU_1/a.$

From this division, we get

$$(b+V_1X)/(a+U_1X) = b/a + (WX)/(a+U_1X),$$

or, by (9),

$$V/U = b/a + (WX)/U.$$

On dividing through by X^n, we get

(11) $\qquad V/[X^n] + W/[X^{n-1}U],$

where W is given by (10).

The process is now repeated, the object of consideration being the fraction $W/[X^{n-1}U]$ after it has been reduced to its lowest terms by striking out any factors common to W and X^n. (It is evident that W and U have no common factors, for if they had, it would follow from (11) that V and U would have the same common factors.)

We note in passing that, by (9), a and b are respectively the values of U and V for that value of x for which X vanishes. This observation justifies a common method for obtaining the partial fractions in the case in which the denominator of the original fraction consists of linear factors only, none of which are repeated.

2. *Quadratic factors.* We turn now to the case in which the denominator contains two or more quadratic factors, and illustrate the method by the consideration of the fraction

$$(x^7+x^2+x+2)/[(x^2+x+1)^3(x^2+2x+2)^2].$$

Set $X = x^2+x+1$. We then find by aid of two divisions

(12) $\qquad x^7+x^2+x+2 = (x+1) + (x^5 - x^4 + x^2 - x + 1)X$

$\qquad\qquad (x^2+2x+2)^2 = x + (x^2+3x+4)X$

We next perform the following "division":

$$\begin{array}{l|r}
 & -x \\
\hline
x+(x^2+3x+4)X & (x+1)+(x^5-x^4+ox^3+1x^2-1x+1)X \\
 & (x+1)+(\qquad\qquad -x^3-3x^2-4x-1)X \\
\hline
 & (x^5-x^4+1x^3+4x^2+3x+2)X
\end{array}$$

This division process requires some explanation. In the first place the term $-x$ in the quotient is obtained as follows. It is first represented by $Ax+B$. The unknowns A, B are then

determined by setting $x(Ax+B)$—AX or $(B—A)x$—A equal to $x+1$. This gives A=—1, B=o. The next step requires that $x+(x^2+3x+4)X$ be multiplied by $-x$. This may conveniently be done in the following form.

$$\begin{array}{ll}
x & +(\quad x^2+3x+4)X \\
-x & \\
\hline
(-x^2\quad) & +(-x^3-3x^2-4x\quad)X \\
(x^2+x+1) & +(\qquad\qquad -1)X \\
\hline
(x+1) & +(-x^3-3x^2-4x-1)X
\end{array}$$

Here the only novel feature is the introduction of the vanishing line $(x^2+x+1)+(\quad -1)X$ to reduce the first term of the product to a linear expression.*

From the above "division" it follows that

$$[(x+1)+(x^5-x^4+x^2-x+1)X]/[x+(x^2+3x+4)X]$$
$$=-x+[(x^5-x^4+x^3+4x^2+3x+2)X]/[x+(x^2+3x+4)X]$$

or, by (12),

$$(x^7+x^2+x+2)/(x^2+2x+2)^2=-x$$
$$+[(x^5-x^4+x^3+4x^2+3x+2)(x^2+x+1)]/(x^2+2x+2)^2$$

On dividing through by $(x^2+x+1)^3$, we get

(13) $$(x^7+x^2+x+2)/[(x^2+x+1)^3(x^2+2x+2)^2]=-x/(x^2+x+1)^3$$
$$+(x^5-x^4+x^3+4x^2+3x+2)/[(x^2+x+1)^2(x^2+2x+2)^2].$$

Similarly by two successive steps, we get

(14) $$(x^5-x^4+x^3+4x^2+3x+2)/[(x^2+x+1)^2(x^2+2x+2)^2]$$
$$=(2x-1)/(x^2+x+1)^2+(-x^3-7x^2-3x+6)/[(x^2+x+1)(x^2+2x+2)^2],$$

(15) $$(-x^3-7x^2-3x+6)/[(x^2+x+1)(x^2+2x+2)^2]=(-12x-8)/(x^2+x+1)$$
$$+(12x^3+44x^2+71x+38)/(x^2+2x+2)^2.$$

*The reader should compare this with the following form for adding angles.

18°	$36'$
20°	$43'$
38°	$791'$
1°—	$60'$
39°	$19'$

Also

$$12x^3+44x^2+71x+38=(7x-2)+(12x+20)(x^2+2x+2).$$

Hence

(16) $$(12x^3+44x^2+71x+38)/(x^2+2x+2)^2=(7x-2)/(x^2+2x+2)^2 + (12x+20)/(x^2+2x+2).$$

From (13), (14), (15), (16) the required result follows.

Let us now describe the above process in general terms. Let the fraction to be resolved be $V/(X^nU)$, where X, U, V are polynomials in x mutually prime and X is a quadratic, say $X=x^2+px+q$. By two divisions, we get

(17) $$U=(a_0x+a_1)+U_1X, \quad V=(b_0x+b_1)+V_1X.$$

We now show that

(18) $$a_1^2-pa_1a_0+qa_0^2 \neq o.$$

Suppose first that $a_0=o$. In that case $a_1 \neq o$, since otherwise it would follow from the first of (17) that X would be a factor of U. Hence (18) holds if $a_0=o$. Suppose $a_0 \neq o$. Now a_0x+a_1 is not a factor of X; for if it were it would follow from the first equation of (17) that it would be a common factor of X and U. Hence the remainder obtained by dividing X by a_0x+a_1 is not zero. But by the remainder theorem this is $(-a_1/a_0)^2+p(-a_1/a_0)+q$ or $(a_1^2-pa_1a_0+qa_0)^2$. Hence (18) is true if $a_0 \neq o$.

We have the following division.

$$\begin{array}{r|l} & Ax+B \\ \hline (a_0x+a_1)+U_1X & (b_0x+b_1)+V_1X \\ & \underline{(b_0x+b_1)+ZX} \\ & \qquad\qquad WX \end{array}$$

Here

(19) $$W=V_1-Z$$

Also A and B are found by setting $(a_0x+a_1)(Ax+B)-a_0AX$ equal to b_0x+b_1. But

$$(a_0x+a_1)(Ax+B)-a_0AX$$
$$=[(a_1-pa_0)A+a_0B]x+[-qa_0A+a_1B]$$

Hence A and B are determined by the system

(20) $$(a_1-pa_0)A+a_0B=b_0$$
$$-qa_0A+a_1B=b_1$$

By (18) this system has a unique solution. Finally Z is found by multiplying $(a_0x+a_1)+U_1X$ by $Ax+B$. A convement form for this is the following.

$$\begin{array}{llcl}
(a_0x+a_1) & & + & U_1X \\
(Ax+B) & & & \\
\hline
(a_0Ax^2+a_1Ax &) & + & (AxU_1)X \\
(& a_0Bx+a_1B) & + & (BU_1)X \\
(-a_0Ax^2-pa_0Ax & -qa_0A) & + & (a_0A)X \\
\hline
 & (b_0x+b_1) & + & Z \quad X
\end{array}$$

The only novelty in this is the introduction of the vanishing line $(-a_0A^2x-pa_0Ax-qa_0A)+a_0A)X$ for the purpose of reducing the first term of the product to a linear function. We conclude from this multiplication that

(21) $\qquad Z=AxU_1+BU_1+a_0A.$

From the above division we obtain

$$[(b_0x+b_1)+V_1X]/[(a_0x+a_1)+U_1X]=(Ax+B)$$
$$+(WX)/[(a_0x+a_1)+U_1X],$$

or, by (17),

$$V/U=(Ax+B)+(WX)/U.$$

On dividing both members of this by X^n, we get

(22) $\qquad V/(X^nU)=(Ax+B)/X^n+W/(X^{n-1}U).$

The process is then repeated with reference to the fraction $W/(X^{n-1}U)$ after it has been reduced to its lowest terms (if necessary). Ultimately we are led to a fraction of the form V/X^n. This is treated by means of the formula.

(22) $\qquad V/X^n=(b_0x+b_1)/X^n+U_1/X^{n-1},$

which results from dividing by X^n both members of the second equation of (17). In this way the resolution is completed.

3. *Conclusion.* It is believed that the above method is the shortest one in the case of difficult examples. If it does not seem so to the reader let him solve by the usual method the second example solved above. He will have to obtain and solve ten equations in ten unknowns. The advantage of our method is particularly great if the work is carried out with detached coefficients.